TIMID PLAY FOR DISCRETE, SUPERFAIR GAMBLING PROBLEMS

by

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Technical Report No. 174

March 1972

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*This research was supported by the National Science Foundation under NSF Grant GP 24183.

1. Introduction.

Let n be a positive integer. Suppose the goal of a gambler is to reach or exceed n. If the gambler has fortune k, a positive integer, he may stake any integral amount s such that $0 \leq s \leq k$. Also, if he stakes s, he wins rs with probability p and loses s with probability $q = 1 - p$. Here r is a fixed positive integer and p a fixed number between 0 and 1. Assuming there is no time limit so that the gambler may continue playing until his fortune is either 0 or at least n, how should the gambler play so as to maximize his probability of reaching n?

Dubins and Savage ([3], Chapter 6) considered a continuous version of the same problem (the gambler being allowed to stake any non-negative real number less than his fortune) and showed that, if the game is subfair in the sense that $pr - q \leq 0$, then the gambler should play boldly and, at each play, stake either his entire fortune or just enough to reach the goal whichever is smaller.

Assume for the remainder of this note that the game is superfair in the sense that $pr - q \geq 0$. By timid play is meant the strategy of always staking 1 at a fortune k if $0 < k < n$ and staking 0 otherwise. In the next section timid play is shown to be optimal under the assumption of superfairness. Further evidence that small bets are good in superfair gambling problems with no bound on playing time may be found in [1], Chapter 10 of [3], [5], and [6].

2. Timid Play is Optimal.

Let $T_n(k)$ denote the probability of reaching n for a timid gambler starting from k. Suppose a gambler with fortune k first stakes s and plays timidly thereafter. Then his probability of reaching n would be

$$B_n(s,\ k) = pT_n(k+rs) + qT_n(k-s).$$

If timid play is optimal, then it must be the case that

$$B_n(s,\ k) \leq T_n(k) \tag{1}$$

for all $k = 0,1,\ldots$ and $s = 0,1,\ldots,\ k$. The fundamental theorem of gambling (Theorem 2.12.1, [3]) implies that the converse is true. That is, to prove timid play is optimal, it suffices to check (1).

The proof is by induction on n. Timid play is trivially optimal for $n = 1$. So assume $n > 1$ and that timid play is optimal for goals smaller than n.

Notice that $T_n(k)$ is the probability that a random walk starting from k which moves one unit to the left with probability q or r units to the right with probability p at each step reaches or exceeds n before it reaches 0. To reach n before 0 the walk must either reach n before 1 or reach 1 before n and then, starting from 1, go on to reach n before 0. It follows that, for $k > 0$,

$$\begin{aligned} T_n(k) &= T_{n-1}(k-1) + [1 - T_{n-1}(k-1)]T_n(1) \\ &= [1 - T_n(1)]T_{n-1}(k-1) + T_n(1) \end{aligned} \tag{2}$$

and so

$$B_n(s,k) = [1 - T_n(1)]B_{n-1}(s,\ k-1) + T_n(1). \tag{3}$$

Now, if $0 \leq s \leq k - 1$, then, by the inductive hypothesis, $B_{n-1}(s,\ k-1)$ $\leq T_{n-1}(k-1)$, which, together with (2) and (3), implies the inequality in (1).

It remains only to check (1) in the case of <u>bold bets</u>; that is, when $s = k$. In other words, it is sufficient to check that timid play is superior to an initial bold bet followed by timid play. Now a gambler starting at k who stakes k initially has 0 with probability q and $k(1+r)$ with probability p after the first bet. Set $m = k(1+r)$. Suppose $m < n$. Then, by the inductive hypothesis, timid play is optimal if the goal is m. Therefore, the chance of reaching or exceeding m for a timid player starting from k must be at least p. After attaining m, both strategies under consideration continue with timid play and so the timid player has at least as great conditional probability of going on to reach n. The desired inequality follows. Now suppose $m \geq n$. In this case (1) becomes $p \leq T_n(k)$. Since $T_m(k) \leq T_n(k)$, it certainly suffices to show $p \leq T_m(k)$. Let $k, X_1, X_2, \ldots$ be the sequence of random variables corresponding to the sequence of fortunes experienced by a gambler who starts at k and plays timidly in a game whose goal is m. By the assumption of superfairness, this sequence is an expectation increasing process or a submartingale. Let t be the time at which the gambler first reaches 0 or m. That is,

$$t = \min\{i : X_i = 0 \text{ or } X_i \geq m\}.$$

Then $t \geq k$ and one can invoke either a stopping time theorem for submartingales or Wald's equation (for a discussion of both these results and further references, see [2]) to conclude that $EX_t \geq EX_k = k + k(pr-q) = pm$. On the other hand, $EX_t \leq mP[X_t \geq m]$, and $P[X_t \geq m] = T_m(k)$. Hence, $T_m(k) \geq p$ and the proof that timid play is optimal is complete.

Suppose now that $pr - q > 0$ and $p < 1$. It is possible to imitate the argument above and show that the inequality in (1) is strict for $1 < k < n$ and $1 < s \leq k$. This implies the unique optimality of timid bets at fortunes between 1 and n.

3. Red-and-Black.

If $r = 1$, then the gambling problem under consideration is called red-and-black. Also, it is a famous result known as "the gambler's ruin" ([4], p. 313) that, for $0 < k < n$ and $p \neq 1/2$,

$$T_n(k) = \frac{1 - (q/p)^k}{1 - (q/p)^n} .$$

This exact expression makes possible a simple alternative proof of (1) for this case.

4. Remarks.

Two extensions of the result just proved naturally suggest themselves. Consider a superfair betting situation in which there are more than two possible outcomes at each stage. For example, if the gambler stakes s, he might lose the stake with probability q or win $r_i s$ with probability p_i for $i = 1, \ldots, \ell$. Unfortunately, the situation here is much more complicated and the optimal strategy for reaching a goal need not be timid even if $\ell = 2$. Another natural generalization is to assume the same setting as in section 1 except that there is a fixed minimum stake $m > 1$. If the gambler's fortune is less than m, he has lost the game. A naive conjecture is that the optimal strategy is always to stake m until either the goal is reached or the gambler has less than m. The conjecture is easily disproved and the optimal strategy is not known. By the way, it is equally natural to consider subfair problems with a maximum stake. Recent results on such games are in [7] and [8].

Acknowledgement

Thanks are due to Anthony Fanning for help with the original proof, to David Gilat for help with the proof given here, and to David Heath for help in finding counterexamples to the suggested extensions of section 4.

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